Solutions: Homework 1

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Problem 1. If (X, d) is any metric space show that every open ball is, in fact, an open set. Also, show that every closed ball is a closed set.

Proof. Let $U = B(x; r)$ denote the open ball in X with center x and radius r. Let $y \in U$ and $d(x, y) = r - \epsilon$ for some $0 < \epsilon \leq r$. Then an application of the triangle inequality shows that $B(y; \epsilon) \in U$. So U is open. Now, let $V = \overline{B}(x; r)$ denote the closed ball in X with center x and radius r. We will show that V^c is open. Let $y \in V^c$ and $d(x, y) = r + \epsilon$ for some $\epsilon > 0$. Then $B(y; \epsilon) \in V^c$ by the triangle inequality. So V^c is open, hence V is closed. □

Problem 2. Let (X, d) be a metric space. Then:
(a) The sets X and ϕ are closed;
(b) If $F_1, ..., F_n$ are closed sets in X then so is $\cup_{k=1}^n F_k$;
(c) If $\{F_j : j \in J\}$ is any collection of closed sets in X, J any indexing set, then $F = \cap\{F_j : j \in J\}$ is also closed.

Proof. (a) $X^c = \phi$ and $\phi^c = X$ and since ϕ and X are open, they should also be closed.

(b) By definition, $F_1^c, ..., F_n^c$ are open sets in X. Since finite intersections of open sets are open, $\cap_{k=1}^n F_k^c$ is open. So $\cup_{k=1}^n F_k = (\cap_{k=1}^n F_k^c)^c$ is closed.

(c) Again, by definition, we have $\{F_j^c : j \in J\}$ is an arbitrary collection of open sets in X. Since arbitrary unions of open sets are open, $F^c = \cup\{F_j^c : j \in J\}$ is open. So F is closed. □

Problem 3. Show that $(\mathbb{C}_\infty, d)$ where d is given by

$$d(z, z') = \frac{2|z - z'|}{[(1 + |z|^2)(1 + |z'|^2)]^{\frac{1}{2}}} \qquad (z, z' \in \mathbb{C})$$

and

$$d(z, \infty) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}$$

is a metric space.

Proof. Following the notation in §6 of Chapter 1, let $f : \mathbb{C}_\infty \to S$ denote the inverse of the stereographic projection. Let dist denote the Euclidean metric on $\mathbb{R}^3$, restricted to S in this case. Then we know that $d(z, z') = \text{dist}(f(z), f(z'))$, by construction. Since dist is a metric, $d(z, z') = \text{dist}(f(z), f(z')) \geq 0$. Suppose $d(z, z') = 0$. Then $\text{dist}(f(z), f(z')) = 0$, hence $f(z) = f(z')$, and so $z = z'$ as f is bijective. Again, the symmetry and triangle inequality just carries over from dist to d. So d is a metric. □

Problem 4. Let A and B be subsets of a metric space (X, d). Then:
(a) A is open if and only if $A = \text{int } A$;
(b) A is closed if and only if $A = \overline{A}$;
(c) $\text{int } A = X - \overline{(X - A)}; \overline{A} = X - \text{int}(X - A); \partial A = \overline{A} - \text{int } A$;
(d) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$;
(e) $x_0 \in \text{int } A$ if and only if there is an $\epsilon > 0$ such that $B(x_0; \epsilon) \subset A$.

Proof. (a) Since int A is open, if int $A = A$, then A is open. Conversely, if A is open, then A is an element of $\{G : G \text{ is open and } G \subset A\}$. So int $A = A$.

(b) Since $\overline{A}$ is closed, if $\overline{A} = A$, then A is closed. Conversely, if A is closed, then A is an element of $\{F : F \text{ is closed and } F \supset A\}$. So $\overline{A} = A$.

(c) Since int $A \subset A$, we have $X - A \subset X -$ int A. Since $X -$ int A is closed, we have $\overline{(X - A)} \subset X -$ int A. Similarly, since $X - A \subset \overline{(X - A)}$, so $X - \overline{(X - A)} \subset A$. Since $X - \overline{(X - A)}$ is open, we have $X - \overline{(X - A)} \subset$ int A. So $X -$ int $A \subset \overline{(X - A)}$. This gives us the first equality. Now if we replace A by $X - A$ in the first one, we get the second one. Now $\overline{A} -$ int $A = \overline{A} \cap (X -$ int $A) = \overline{A} \cap \overline{(X - A)}$ (by the first equality) $= \partial A$.

(d) Since $A \subset \overline{A}$ and $B \subset \overline{B}$, we have $A \cup B \subset \overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B}$ is closed, we have $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Now $A \subset A \cup B$, so $\overline{A} \subset \overline{(A \cup B)}$. Similarly, $\overline{B} \subset \overline{(A \cup B)}$. So $\overline{A} \cup \overline{B} \subset \overline{(A \cup B)}$. This gives equality.

(e) If $x_0 \in$ int $A, \exists \epsilon > 0$ such that $B(x_0; \epsilon) \subset$ int A, as int A is open. Since int $A \subset A$, we have $B(x_0; \epsilon) \subset A$. On the other hand, if $\exists \epsilon > 0$ such that $B(x_0; \epsilon) \subset A$, then since $B(x_0; \epsilon)$ is open, $B(x_0; \epsilon) \subset$ int A, hence $x_0 \in A$. □

Problem 5. The purpose of this exercise is to show that a connected subset of $\mathbb{R}$ is an interval.
(a) Show that a set $A \subset \mathbb{R}$ is an interval iff for any two points a and b in A with $a < b$, the interval $[a, b] \subset A$.
(b) Use part (a) to show that if a set $A \subset \mathbb{R}$ is connected then it is an interval.

Proof. (a) It is clear that if A is an interval, then the given criterion holds. Conversely, suppose this holds. Let $c = \inf A$ and $d = \sup A$. Note that these could be $-\infty$ and $+\infty$ respectively. If $c = d$, it is clear that $A = \{c\}$, which is an interval. So, suppose $c < d$. We claim that the interval $(c, d) \subset A$. Suppose not. Then $\exists x \in (c, d)$ such that $x \notin A$. Since x

is neither the supremum, nor the infimum of $A, \exists s < x < t$ with $s, t \in A$. But by the given criterion, $[s, t] \subset A$. In particular, $x \in A$, a contradiction. So $(c, d) \subset A$. Now depending on whether c and/or d are contained in A, we see that A is either an open, semi-open or closed interval.

(b) Suppose A is connected, but not an interval. Then, by part (a), $\exists a, b \in A$ such that $\exists x \in [a, b]$, but $x \notin A$. Then the set $A \cap (-\infty, x] = A \cap (-\infty, x)$ is a proper, non-empty subset of A that is both open and closed. This contradicts the fact that A is connected. So A is an interval. □

Problem 6. Prove that if $\{D_j : j \in J\}$ is a collection of connected subsets of X and if for each j and k in J we have $D_j \cap D_k \neq \phi$ then $D = \cup\{D_j : j \in J\}$ is connected.

Proof. Suppose that D is not connected. Choose a non-empty proper subset A of D that is both open and closed. Since the $D_j's$ are connected and $A \cap D_j$ is both open and closed in D_j, we get that $A \cap D_j = \phi$ or D_j. Choose j and k in J such that $A \cap D_j = D_j$ and $A \cap D_k = \phi$. Note that we can find such a pair j and k because otherwise, we either have $D_j \subset A$ $\forall j \in J$ or $D_j \subset D - A$ $\forall j \in J$, which would imply that $A = D$ or $A = \phi$ respectively. Then $(D - A) \cap D_j = \phi$ and $(D - A) \cap D_k = D_k$. Then $D_j \cap D_k = (A \cap D_k \cap D_j) \cup ((D - A) \cap D_j \cap D_k) = (\phi \cap D_j) \cup (\phi \cap D_k) = \phi$, which contradicts our assumption. □